*Int. J. Solids Structures*, 1973, Vol. 9, pp. 449 to 460. Pergamon Press. Printed in Great Britain

# SHEAR IN $C^0$ AND $C^1$ BENDING FINITE ELEMENTS

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**Abstract**—The Kirchhoff assumption in thin elastic plates results in a biharmonic equation for the lateral deflection and a $C^1$ deflection field is therefore required in the finite element method for their approximate solution. By considering the thin plate as a three dimensional solid and by discarding the Kirchhoff assumption, the continuity requirement for the displacements is reduced to $C^0$. The stiffness matrix produced in this way becomes, however, violently ill-conditioned as the thickness $t$ of the structure is reduced. It is shown here that the factor $1/t^2$ causing this ill-conditioning can be removed from the stiffness matrix and consequently from its condition number by relating the thickness $t$ to the diameter of the element $h$, without losing the rate of convergence provided by the degree of the shape functions inside the element. This is used here to construct a well-conditioned 9-degrees-of-freedom plate bending element which is only $C^0$ but which converges quadratically to the $C^1$ solution (Kirchhoff solution) of thin plates.

Addition of shear to $C^1$ elements is also considered.

## INTRODUCTION

EXTENSIVE effort [1–6] has been invested in constructing bending elements which include shear. This has been done for two purposes: first, for the proper inclusion of shear in thick (sandwich) structures to obtain a more realistic element; secondly for avoiding the continuity of slopes requirement in fourth order (such as bending) problems. By regarding the thin structure as a three-dimensional elastic solid and by discarding the Kirchhoff assumption that normals to the middle surface remain so during bending, the continuity of slopes requirement is replaced by the equivalent (but simpler to satisfy) requirement that both the normal and tangential displacements along the cross sections be continuous between adjacent elements.

Starting with the basic equations of elasticity and performing the various approximations *en route*, rather than starting with a ready shell theory is certainly attractive. However, if shear is added to a $C^0$ element (say an element with a quadratic approximation for the normal displacement inside it) then as the thickness of the structure is being reduced the condition of the global stiffness matrix increases without bound causing numerical difficulties when applied to thin structures.

The cause of this ill-conditioning in the stiffness matrix is the factor $1/t^2$ appearing in it, where $t$ denotes the thickness of the structure. For a Kirchhoff plate the addition of shear can be considered as an error which, when measured in the energy is $0(t^2)$. This means that if the plate is thin there is no use to substitute into the energy expression the true $t$ since the finite element discretization error might be overwhelming. Here the thickness $t$ is related to the element size $h$ such that the shear and discretization errors are balanced. *The thickness t is being reduced as the mesh is being reduced.* In this manner thin, beam, plate and shell finite elements can be constructed *which are only $C^0$ (and hence easy to build)*

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*but which converge to the $C^1$ (Kirchhoff) solution of thin structures.* Moreover, the condition number of the resulting global matrix grows with $h$ like $0(h^{-4})$ [10] as with pure bending ($C^1$) finite elements.

The paper contains an actual derivation and numerical experiments with a 9-degrees-of-freedom triangular plate bending element which is only $C^0$ but which converges quadratically to the solution of thin, shearless plates. A numerical constant in the element stiffness allows also a control over its stiffness (or flexibility).

If the shear is superposed on a $C^1$ element (such as the standard 4-degrees-of-freedom Hermitian beam element) and if the nodal variables are properly chosen and scaled then as the thickness of the structure is being reduced the resulting matrix converges back to the original $C^1$ element without shear; its condition number being that of pure bending matrices.

Additional computational problems resulting from the coupling between the tangential and normal displacements in curved shells are discussed in Ref. [7].

Discretization and round-off errors in eigenproblems are discussed in Refs. [8, 9].

## STRAINS AND ENERGIES

The basic ideas of this paper will be first explained on explicit one dimensional beam problems.

Let $\phi$, $w$ and $u$ denote the rotation of the cross section with respect to the middle line (with the Kirchhoff assumption $\phi = 0$), the normal displacement of the middle line and the tangential displacement of the cross section, respectively. Assume also that there is no normal displacement relative to the middle line. It is simplest to assume that the cross section remains straight during bending. Then

$$u = -y\left(\phi+\frac{\mathrm{d}w}{\mathrm{d}x}\right) \tag{1}$$

where $x$ denotes the coordinate along the middle line and $y$ along the normal to it. With equation (1) the shear strain is just $\phi$, since

$$-\left(\frac{\mathrm{d}u}{\mathrm{d}y}+\frac{\mathrm{d}w}{\mathrm{d}x}\right) = \phi. \tag{2}$$

The assumption that the shear is constant along the normal introduces a relative error of $0(t^2)$ in the total elastic energy and an error $0(1)$ in the shear energy, where $t$ denotes the thickness of the structure. An obvious way to improve the accuracy of the results is to provide for a better approximation of the shear along the normal with more degrees of freedom. A more realistic variation of $\phi$ along the normal without increasing the number of nodal degrees of freedom may also be expected to improve the accuracy, at least in the interior. For instance, the shear can be varied quadratically along the normal, vanishing on the surfaces, but made to depend on *only one degree of freedom* $\phi$. Thus

$$\frac{\mathrm{d}u}{\mathrm{d}y}+\frac{\mathrm{d}w}{\mathrm{d}x} = -\phi\left(1-4\frac{y^2}{t^2}\right) \tag{3}$$

where $-t/2 \leq y \leq t/2$ and hence

$$u = -\phi\left(y-\frac{4}{3}\frac{y^3}{t^2}\right)-\frac{\mathrm{d}w}{\mathrm{d}x}y. \tag{4}$$

In a laminated thin structure the shear can again be made to depend on only 1-degree-of-freedom but to vary along the cross section in accordance with the variation of the elastic properties of each layer.

More details about the stress distribution near boundaries, point forces or corners can be obtained with a local fine mesh of spatial elements.

For a linear variation of $u$ along the cross section, the direct strain $e$ becomes

$$e = -y\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}+\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) \tag{5}$$

and the bending energy $U_b$ for a beam of unit length is therefore of the form

$$U_b = \frac{1}{2}\frac{Et^3}{12}\int_0^1\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}+\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2\mathrm{d}x \tag{6}$$

while the shear energy $U_s$ is of the form

$$U_s = \tfrac{1}{2}Gt\int_0^1\phi^2\,\mathrm{d}x \tag{7}$$

where $E$ and $G$ denote the elastic tension and shear moduli. The total energy $U = U_b+U_s$ can be written as

$$U = \frac{1}{2}\frac{Et^3}{12}\left[\int_0^1\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}+\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2\mathrm{d}x+\frac{12G}{E}\frac{1}{t^2}\int_0^1\phi^2\,\mathrm{d}x\right]. \tag{8}$$

In the variational formulation of the elastic problem the total potential energy is minimized. The ever increasing factor $1/t^2$ before the second integral in equation (8) means then that as $t$ tends to zero, $\phi$ is also forced to zero. In the finite element analysis, the introduction of shear in thin bending elements can be interpreted as *enforcing a $C^1$ continuity in the least squares sense.*

## $C^0$ ELEMENTS

Elements with which it is impossible to assure the continuity of slopes by just assembling them together are termed $C^0$ elements. The computational aspects of the inclusion of shear in such elements is studied in this section on an element with three nodal points between which the normal displacement $w$ is interpolated quadratically. Obviously this element cannot be used directly in bending problems since only $w$ but not $\mathrm{d}w/\mathrm{d}x$ is continuous across elements. To this element shear is added, therefore, through two additional degrees of freedom $\phi_1$ and $\phi_2$ at the extremities and the shear $\phi$ is interpolated linearly between them. The element is associated now with the 5-degrees-of-freedom $w_1$, $\phi_1$, $w_3$, $w_2$, $\phi_2$ and is, in fact, the one dimensional version of the 9-nodal-point shell element discussed in Ref. [17]. For insuring the continuity of both $u$ and $w$ across the elements, $\phi$ is replaced by the *total rotation* $\theta = \mathrm{d}w/\mathrm{d}x+\phi$ and the element becomes associated with the nodal degrees of freedom $w_1$, $\theta_1$, $w_3$, $w_2$, $\theta_2$.

Let $h$ denote the length of the element. With $x = h\xi$ the interpolation formula for $w$ reads

$$w = w_1(2\xi^2 - 3\xi + 1) + w_2(2\xi^2 - \xi) + w_3 4(\xi - \xi^2) \tag{9}$$

while that for $\phi$ reads

$$\phi = \phi_1(1 - \xi) + \phi_2 \xi. \tag{10}$$

From equations (9) and (10) and with $\theta = \mathrm{d}w/\mathrm{d}x + \phi$ it results that

$$\begin{aligned} \phi_1 &= (-3w_1 - w_2 + 4w_3 - h\theta_1)/h \\ \phi_2 &= (w_1 + 3w_2 - 4w_3 - h\theta_2)/h. \end{aligned} \tag{11}$$

Introducing $\phi$ and $w$ as given by equations (9) and (10) into the energy expression (8), yields, together with equations (11), the element stiffness matrix $k$ in the form

$$k = \frac{EI}{h^3}\left\{ \begin{bmatrix} & & & & \\ & 1 & & & -1 \\ & & & & \\ & & & & \\ & -1 & & & 1 \end{bmatrix} + \frac{2G}{E}\left(\frac{h}{t}\right)^2 \begin{bmatrix} 14 & 5 & -16 & 2 & 1 \\ 5 & 2 & -4 & -1 & 1 \\ -16 & -4 & 32 & -16 & 4 \\ 2 & -1 & -16 & 14 & -5 \\ 1 & 1 & 4 & -5 & 2 \end{bmatrix} \right\} \tag{12}$$

where $I = t^3/12$. The matrix $k$ in equation (12) refers to the nodal variables $w_1$, $\hat{\theta}_1$, $w_3$, $w_2$, $\hat{\theta}_2$ where $\hat{\theta} = h\theta$. Since $w_3$ is an interior degree of freedom it can be eliminated by "static condensation" (which is in fact a Gauss elimination).

The global stiffness matrix $K$ can be written in the general form

$$K = \frac{EI}{h^3}\left[K_b + \frac{2G}{E}\left(\frac{h}{t}\right)^2 K_s\right] \tag{13}$$

and $K_b$ and $K_s$ include only numbers. Due to the existence in the beam of pure bending modes and pure shearing modes, both $K_b$ and $K_s$ are singular, but not $K$.

The maximum eigenvalue $\lambda_N^K$ of the global stiffness $K$ is readily found to be of the form

$$\lambda_N^K = c_1 + c_2 \frac{2G}{E}\left(\frac{h}{t}\right)^2 \tag{14}$$

where $c_1$ and $c_2$ are independent of $t$ and $h$. For a small $t$ the minimum eigenvalue of $K$, $\lambda_1^K$, is nearly pure flexural and therefore [10, 11]

$$\lambda_1^K = c_3 h^4 \tag{15}$$

where $c_3$ is again independent of $t$ and $h$ (the power 4 in $h$ is, in fact, only approached as the degree of the shape functions increase). Assuming that $2G = E$, the spectral condition number $C_2(K) = \lambda_N^K/\lambda_1^K$ of the global stiffness matrix $K$ becomes

$$C_2(K) = c_4 h^{-4} + c_5(1/t)^2 h^{-2} \tag{16}$$

or

$$C_2(K) = c_4 N_e^4 + c_5(1/t)^2 N_e^2 \tag{17}$$

where $N_e = h^{-1}$ denotes the number of elements in the one dimensional mesh. In two dimensional problems $N_e$ is to be replaced by $N_{es}$—the *number of elements/side* (see the extensive discussion in Refs. [10–12]). The appearance of $1/t^2$ in the expression for $C_2(K)$ in equation (17) causes $C_2(K)$ to grow without bound as $t$ is reduced to zero. For a small $t$ and a small number of elements ($h \gg t$) the second term in equation (16) is much larger than the first one and therefore approximately

$$C_2(K) = c_6(1/t)^2 N_e^2. \tag{18}$$

In fact, numerical calculation yields for a cantilever beam and the element in equation (12)

$$C_2(K) = 10(1/t)^2 N_e^{1\cdot 7}. \tag{19}$$

An expression for $C_2(K)$ similar to the one given in equation was also found in Ref. [19] by a direct calculation of the extremal eigenvalues of $K$.

According to Bauer [13, 14] the round-off error $\|\delta x\|_2 = (\delta x^T \delta x)^{1/2}$, committed in solving the linear system $Kx = b$ on a computer with $s$ decimals is given by

$$\frac{\|\delta x\|_2}{\|x\|_2} = C_2(K)10^{-s}. \tag{20}$$

Hence a high condition number spells numerical trouble and for $C_2(K)10^{-s} \geq 1$ the matrix $K$ is practically singular. Singularity in the stiffness matrix just established is quick to come. It becomes singular when

$$10^{-s}10(1/t^2)N_e^{1\cdot 7} = 1 \tag{21}$$

is reached. On a computer with 7 decimals and with $t = 1/100$, the stiffness matrix assembled over 10 elements is already singular.

It will be shown now that *the factor $1/t^2$ can be removed from the element stiffness matrix in equation* (12) *and hence from $C_2(K)$ in equation* (17) *while retaining the full accuracy provided by the shape functions.* The resulting $C^0$ element thus constructed will be applicable to shearless (Kirchhoff) thin structures.

If $w$ is interpolated inside the element by a complete polynomial of degree $p$ then the error in the energy [15, 16] for the *shearless* structure is $0(h^{2(p-1)})$ (one has to notice for this that if the *exact* solution is a polynomial of degree $p$ then the finite element exactly fits this $C^1$ surface). For the 3-nodal-point element of this section $p = 2$ and the error in the pure bending energy is $0(h^2)$. Shear changes the energy by $0(t^2)$. Therefore, if the beam is very thin, introducing the exact value for $t$ into the stiffness matrix in equation (12) is not justified if the discretization errors are still large. With a small number of elements the discretization error [which is $0(h^{2(p-1)})$] might be much larger than the shear effect [which is $0(t^2)$]. To fix ideas consider a cantilever beam with a tip force. With one element the tip deflection is given by

$$\text{tip deflection} = \tfrac{1}{6}t^2 + \tfrac{1}{4} \tag{22}$$

the exact tip deflection for a shearless beam being $\frac{1}{3}$. If $t$ is set equal to zero in equation (22) the tip deflection becomes $\frac{1}{4}$ and the discretization error is $\frac{1}{12}$. If $t$ is set equal to 1 in equation (22) the tip deflection becomes $\frac{5}{12}$ with an error equal to $-\frac{1}{12}$, and for $t = 0$ and $t = 1$ the errors are comparable. *The best value for t to be introduced into the element matrix is that for which the discretization error and the shear effect are balanced.* This occurs with

$$t^2 = \frac{1}{c} h^{2(p-1)} \tag{23}$$

where $c$ is a proportionality coefficient. For the quadratic ($p = 2$) element of this section $t = (1/c)h$ and with the arbitrary choice of $c = 1$ the element stiffness matrix $k$ in equation (12) becomes

$$k = \frac{EI}{h^3} \begin{bmatrix} 14 & 5 & -16 & 2 & 1 \\ 5 & 3 & -4 & -1 & 0 \\ -6 & -4 & 32 & -16 & 4 \\ 2 & -1 & -16 & 14 & -5 \\ 1 & 0 & 4 & -5 & 3 \end{bmatrix} \tag{24}$$

This is a $C^0$ element (non Kirchhoff) but it converges to the solution of a shearless (Kirchhoff) beam with an energy rate of convergence $0(h^2)$, while its condition number is $0(h^{-4})$ as with $C^1$ bending elements.

The accuracy of the tip deflection in a cantilever beam discretized with the element matrix $k$ in equation (24) was tested also numerically and it was found that

$$\text{relative error in tip deflection} = 0{\cdot}4h^2 \tag{25}$$

verifying that this element is of quadratic accuracy. It was also found numerically that in this case $C_2(K) = 50\,h^{-3{\cdot}2}$.

The coefficient $c$ in equation (23) controls the flexibility of the element. If $c$ is *decreased* the matrix becomes more *flexible* (tip deflection increases). An optimal $c$ can be selected based on numerical experiments. An exact tip deflection, for instance, is obtained with $c = \sqrt{2}$.

A similar control over the flexibility of the stiffness matrix is claimed by the "reduced integration technique" [17, 18].

## $C^1$ ELEMENTS

An element assuring continuity of both displacements and slopes is termed a $C^1$ element. The simplest in one dimension is with the four nodal values $w_1$, $w_{x1}$, $w_2$, $w_{x2}$ ($w_x = \mathrm{d}w/\mathrm{d}x$). Shear is superposed on this element by adding to two degrees of freedom $\phi_1$ and $\phi_2$ at the ends and interpolating $\phi$ linearly. It is easily shown that if $\mathrm{d}w/\mathrm{d}x$ is replaced by the total rotation $\theta = \mathrm{d}w/\mathrm{d}x + \phi$, then the scaling

$$\phi = \hat{\phi}t/h^2 \tag{26}$$

avoids $1/t^2$ in $C_2(K)$. Indeed, the element stiffness matrix $k$ in this case is given by (with $2G = E$)

$$k = \frac{EI}{h^3}\left(\begin{array}{ccc|ccc} 12 & 6 & 6\tau & -12 & 6 & 6\tau \\ \hline 6 & 4 & 3\tau & 6 & 2 & 3 \\ \hline 6\tau & 3\tau & 3\tau^2+2 & -6\tau & 3\tau & 3\tau^2+1 \\ \hline -12 & -6 & -6\tau & 12 & -6 & -6\tau \\ \hline 6 & 2 & 3 & -6 & 4 & 3\tau \\ \hline 6\tau & 3\tau & 3\tau^2+1 & -6\tau & 3\tau & 3\tau^2+2 \end{array}\right) \tag{27}$$

where $\tau = h/t$. The stiffness matrix in equation (27) refers to the nodal variables $w_1$, $\hat{\theta}_1$, $\hat{\phi}_1$, $w_2$, $\hat{\theta}_2$, $\hat{\phi}_2$. Evidently, as $t$ is reduced to zero, $k$ in equation (27) converges back to the stiffness of the standard beam element plus the condition $\phi = 0$. In the case of shearless beam and plate problem the condition number of the global stiffness matrix $K$ is of the general form [11, 15]

$$C_2(K) = c_7 N_{es}^4 \tag{28}$$

where $N_{es}$ denotes the number of elements/side and where $c_7$ is independent of $N_{es}$.

Since the coupling of the shear $\phi$ (or $\hat{\phi}$) between the elements is optional, $\phi_1$ and $\phi_2$ can be eliminated from $k$ by "static condensation".

It should be well noticed that the elimination of $1/t^2$ from $C_2(K)$ is due to the particular choice of the nodal variables $\theta$—the total rotation and $\phi$—the shear. This would not have been possible with the nodal variables $\theta$ and $w_x$.

## TRIANGULAR AND RECTANGULAR BENDING ELEMENTS

Let the elements lie in the $x$, $y$ plane such that $z$ is normal to the middle surface of the plate. We concentrate in this section on the simplest plate bending element—a triangle with the 9-degrees-of-freedom $w_1$, $w_{x1}$, $w_{y1}$, $w_{x2}$, $w_{y2}$, $w_3$, $w_{x3}$, $w_{y3}$ at the vertices, where $w_x = \partial w/\partial x$ and $w_y = \partial w/\partial y$. This element has the same degrees of freedom as the $C^1$ Hsieh–Clough–Tocher (HCT) element [20] obtained by dividing the element into three sub-triangles. The derivation of its companion 12-degrees-of-freedom rectangular element with $w$, $w_x$ and $w_y$ at the corners is completely analogous to that of the triangular element.

The shape functions for the lateral deflection $w$ inside our triangular element include a complete polynomial of the second degree and for an element with vertices at (0, 0), (1, 0), (0, 1) they are $1, x, y, x^2, xy, y^2, x^3, x^2y-xy^2, y^3$. With this, $w$ varies cubically along the sides of the element and a $C^0$ continuity is assured for $w$ and for $\partial w/\partial s$ where $s$ is a coordinate parallel to the sides of the elements.

We introduce now the total rotation of the normal to middle surface $\theta_x$ and $\theta_y$ relative to the $x$ and $y$ coordinates, respectively. These rotations are interpolated inside the element in such a way that along the sides of the element $\theta_s = \partial w/\partial s$ (no shear) and also such that the rotation $\theta_n$ normal to the sides of the element varies linearly along the sides. This implies in particular that *at the nodal points*, $\theta_x = \partial w/\partial x$ and $\theta_y = \partial w/\partial y$ and the introduction of $\theta_x$ and $\theta_y$ can hence be interpreted as an *independent* interpolation of $w$, $\partial w/\partial x$ and $\partial w/\partial y$. Obviously, the way we interpolated $\theta_x$ and $\theta_y$ assures their continuity across the boundaries of the elements.

We assume the displacements $u$ and $v$ parallel to the middle surface of the plate to vary linearly along $z$. Then

$$u = -\theta_x z \quad \text{and} \quad v = -\theta_y z \tag{29}$$

and the six strain components $e_{xx}$, $e_{yy}$, $e_{zz}$, $e_{xy}$, $e_{xz}$ and $e_{yz}$ become

$$\begin{aligned} e_{xx} &= -\frac{\partial\theta_x}{\partial x}z, \qquad e_{yy} = -\frac{\partial\theta_y}{\partial y}z, \qquad e_{zz} = 0, \\ e_{xy} &= -\left(\frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x}\right)z, \qquad e_{xz} = -\theta_x+\frac{\partial w}{\partial x} \end{aligned} \tag{30}$$

and

$$e_{yz} = -\theta_y+\frac{\partial w}{\partial y}.$$

Following the notation of the first section on strains and energies we have $e_{xz} = -\phi_x$ and $e_{yz} = -\phi_y$.

Let $h$ denote again a typical length in the element (say length of largest side). We transform the element from the $x, y, z$ system to the $\xi, \eta, z$ system where $x = h\xi$ and $y = h\eta$. After performing the integration in the $z$ direction the elastic energy $U$ becomes

$$\begin{aligned} U = \frac{1}{4h^2}\,\frac{Et^3}{12}\Bigg\{&\int_{\Delta'}\left[2\left(\frac{\partial\hat\theta_\xi}{\partial\xi}\right)^2+2\left(\frac{\partial\hat\theta_\eta}{\partial\eta}\right)^2+\left(\frac{\partial\hat\theta_\xi}{\partial\eta}+\frac{\partial\hat\theta_\eta}{\partial\xi}\right)^2\right]\mathrm{d}\xi\,\mathrm{d}\eta \\ &+12\left(\frac{h}{t}\right)^2\int_{\Delta'}\left[\left(\frac{\partial w}{\partial\xi}-\hat\theta_\xi\right)^2+\left(\frac{\partial w}{\partial\eta}-\hat\theta_\eta\right)^2\right]\mathrm{d}\xi\,\mathrm{d}\eta\Bigg\} \end{aligned} \tag{31}$$

where $\Delta'$ denote the limits of integration in $\xi$, $\eta$, where $E$ is the elastic modulus, where $\hat\theta_\xi = \theta_x h$ and $\hat\theta_\eta = \theta_y h$, and where the Poisson ratio was assumed to be zero.

Since the shape functions for $w$ include a complete quadratic and those of $\theta_x$ and $\theta_y$ a complete linear polynomial the asymptotic energy role of convergence of this element is $0(h^2)$. Hence a balanced discretization and shear errors is obtained with $t^2 = h^2/c$ where $c$ is a positive constant. The element stiffness matrix derived from equation (31) with $t^2 = h^2/c$ becomes then of the form

$$k = \frac{1}{2}\,\frac{Et^3}{12}\,\frac{1}{h^2}(k_b+12ck_s) \tag{32}$$

where $k_b$ and $k_s$ are the bending and shear portions. By raising the value of $c$ in equation (32) we can make the element stiffer. In the following numerical experiments we determine the influence of $c$ on the stiffness of the matrix and select the one giving consistently the best results. *One should notice, however, that the asymptotic energy convergence* $0(h^2)$ *is assured for this element independently of the choice of* $c$ *in equation* (33).

Figures 1 and 2 show the results of a static and dynamic experiment with the present triangular element used to discretize a square simply supported plate. Figure 1 shows the

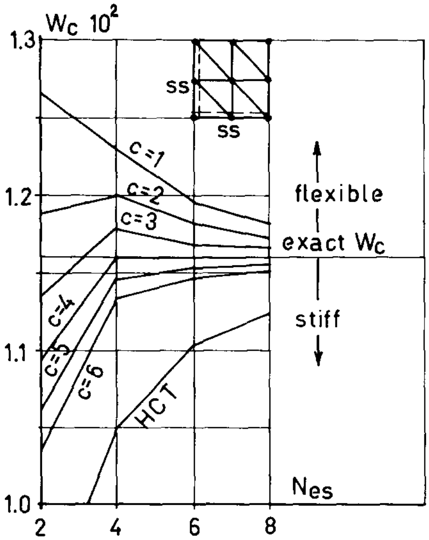


FIG. 1. Convergence of the central deflection $w_c$ vs. the number of elements/side $N_{es}$, in a simply supported square plate acted upon by a point force at the center. The curve HCT refers to discretization with the Hsieh–Clough–Tocher element. The other curves refer to discretization with the present element with different values of the constant $c$ in $t^2 = h^2/c$.

convergence of the central deflection (energy) vs. the number of elements/side $N_{es}$ of the plate acted upon by a central unit point load and with $Et^3/12 = 1$. This central point is a singular point but since the element is $0(h^2)$ and the singularity of the form $r^2 \log r$ the full rate of convergence is obtained [21] with a uniform mesh. It is clearly seen from Fig. 1 that as $c$ in $t^2 = h^2/c$ is increased the element becomes stiffer and for $c > 4$ the energy

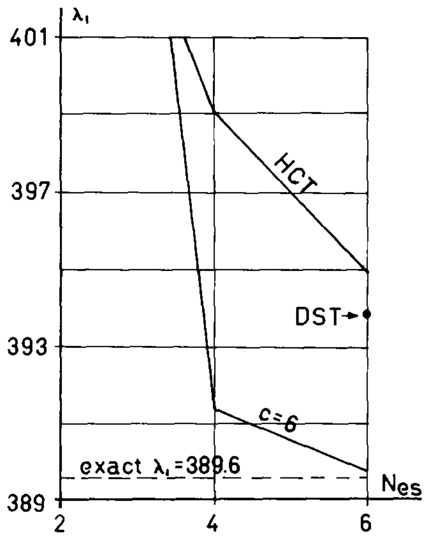


FIG. 2. Convergence of the first eigenvalue $\lambda_1$ of a simply supported plate discretized by the HCT element and by the present element with $t^2 = h^2/6$.

convergence is from above. Figure 1 also shows the convergence of the Hsieh–Clough–Tocher (HCT) element for the same problem. Figure 2 describes the convergence of the first eigenvalue $\lambda_1$ of a simply supported square plate discretized with the present element, the HCT element (results from Ref. [22]) and the 9-degrees-of-freedom element of Dungar, Severn and Taylor (DST) [23] obtained by the assumed stress technique of Pian [24].

The mass matrix for this dynamic analysis was calculated only from $w$ ignoring the rotary inertia.

Calculations were carried out with 14 decimals to minimize the effect of round-off errors.

We conclude from these experiments that $c$ is a reasonable choice for 6.

## CONCLUDING DISCUSSION

When shear is added to a $C^0$ element for use in bending problems the global stiffness matrix becomes of the form

$$K = K_b + \left(\frac{h}{t}\right)^2 K_s \tag{29}$$

where $K_b$ and $K_s$ are the bending and shear portions and where $h/t$ is the diameter to thickness ratio of the element. *This inclusion of shear in thin structures can be regarded as enforcing a $C^1$ continuity (the Kirchhoff assumption) in the least squares sense with a large weighting factor* $1/t^2$. The stiffness matrix in equation (29) is associated with the spectral condition number (ratio between maximum and minimum eigenvalues of $K$).

$$C_2(K) = c_8(1/t)^2 N_{es}^2 \tag{30}$$

where $c_8$ is independent of the thickness $t$ and the number of elements/side $N_{es}$. Equation (30) indicates that as $t$ is being reduced the condition number of the stiffness matrix $K$ increases without bound causing numerical difficulties in thin walled structures.

If the shape functions for $w$ inside the element include a *complete polynomial* of degree $p$ then the error in the bending energy (neglecting shear) is $0(h^{2(p-1)})$. The addition of shear to a shearless structure can be considered as an error term which is $0(t^2)$. *This means that even if the structure is shearless the discretization accuracy does not warrant the introduction of the exact small t into K in equation* (29) *and the full rate of convergence* $0(h^{2(p-1)})$ *(to the Kirchhoff solution) is maintained with*

$$t^2 = \frac{1}{c} h^{2(p-1)} \tag{31}$$

*where c is a proportionality coefficient.* Introducing $t^2$ as given in equation (31) into $K$ in equation (29) results in

$$K = K_b + ch^{-2(p-2)} K_s \tag{32}$$

with the condition number

$$C_2(K) = c_9 N_{es}^4 + c_{10} N_{es}^{2p} \tag{33}$$

*and* $1/t^2$ *is removed from* $C_2(K)$. For the quadratic element discussed in this section $p = 2$ and $C_2(K)$ becomes $0(N_{es}^4)$ as in pure bending problems. For higher order elements (larger $p$) the matrix becomes, with the same number of elements, more ill-conditioned but the discretization accuracy is also higher and one needs less elements for a given accuracy in the results.

The factor $c$ before the shear matrix $K_s$ in equation (32) controls the stiffness of the matrix $K$. Increasing $c$ increases the stiffness of $K$ and the best $c$ yielding the most accurate element can be selected based on numerical computations.

This technique was used here to construct a triangular plate bending element with the three nodal values $w$, $w_x$ and $w_y$ at the vertices. The total rotations $\theta_x$ and $\theta_y$ of the normal to the middle surface where chosen in such a way as to assure a $C^0$ continuity for them but so as to give rise to no shear along the sides of the element. This means that at the nodal points $\theta_x = \partial w/\partial x$ and $\theta_y = \partial w/\partial y$. It also amounts to assuming an *independent* interpolation for $w$, $\partial w/\partial x$ and $\partial w/\partial y$. Since the shape functions for $w$ include a complete quadratic and those for $\theta_x$ and $\theta_y$ a complete linear polynomial, it results that this element can approximate the Kirchhoff energy up to $0(h^2)$. A balanced shear and a discretization error is obtained therefore with $t^2 = h^2/c$ where $c$ is a positive constant. We conclude from the present numerical examples that 6 is a reasonable choice for $c$ [notice that the element assures a quadratic energy convergence $0(h^2)$ to the Kirchhoff solution for any $c$]. These numerical experiments also demonstrate the superior accuracy that can be obtained with this element as compared to a similar $C^1$ element.

The spectral condition number of the global stiffness matrix generated with the present element grows like $0(h^{-4})$ as with $C^1$ bending elements.

It has also been shown that if shear is superposed on a $C^1$ element then with the choice of shear and total rotation as nodal variables and with proper scaling, $1/t^2$ does not appear in $C_2(K)$ which is

$$C_2(K) = c_{11} N_{es}^4 \tag{34}$$

as in pure bending problems.

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(*Received* 12 *June* 1972)

**Абстракт**—Предположение Кирхгоффа в тонких упругих пластинках происходит в результате в бигармоническое уравнение для поперечного изгиба и затем требуется поле прогибов $C^1$ в методе конечного элемента для их приближенных решений. Путем исследования тонкой пластинки, в смысле трехмерного упругого твердого тела и отброса предположения Кирхгоффа, сводится необходимое условие непрерывности для перемещений к $C^0$. Но даже оказывается, что определенная этим способом матрица коэффициентов жесткости находится в очень плохом состоялии, когда толщина т конструкции уменьшается. Определяется здесь что повод этого плохого состояния, то есть отношение $1/\tau^2$ можно удалить из матрицы коэффициентов жесткости и следовательно, из числа их условий, путем соотношения толщины т к диаметру элемента h, без завтраты скорости сходимости, которая является обеспеченная ступенем функций формы внутри элемента. Предлагаемый способ используется, здесь, с целью построения хорошо обусловленного изгибаемого элемента плиты, 9 степени свободы, который только $C^0$, но сходится квадратически к решению $C^1$. Обсуждается также прибавление сдвига к элементам $C^1$.